Math 261B Tues. 9/15

Alg. groups are smooth

Smooth algebraic variety: at $p \in X$ have **regular local coordinates** $x_1, \dots, x_d \in \mathcal{O}(X)$ s.t. $d = \dim X$ and $x_1, \dots, x_d$ generate the ideal m_p of p in the local ring $\mathcal{O}_{X,p}$ of functions on neighborhoods of p.

$$\left(\begin{array}{ccc} \mathcal{O}(X)/m_p & = & K. \\ f & \mapsto & f(p) \end{array} \right)$$

$\rightarrow$ $f \in \mathcal{O}_{X,p}$ is then a polynomial in the x_i times g^{-1} where $g(p) \neq 0$

$\rightarrow$ $\mathcal{O}(X)/m_p^n$ has basis $\underline{x}^{\underline{a}}$ where $|\underline{a}| < n$

degree n

or $\hat{\mathcal{O}}_{X,p} = \varprojlim_n \mathcal{O}(X)/m_p^n \cong K[[x_1, \dots, x_n]]$

Ex. $\underset{\overset{\shortparallel}{p}}{e} = I_n \in GL_n$ $\quad x_{ij} = \underbrace{(X - I)_{ij}}_{\text{general matrix } X \in GL_n}$ generate m_e in $\mathcal{O}(GL_n)$

\# is $n^2 = \dim(GL_n)$

X_{ij}, $z = \det(X)^{-1}$

x_{ij} and $z = \det(X)^{-1}$ gen $\mathcal{O}(GL_n)$

$z(e) = 1 \neq 0.$

A tangent vector at p in X is a point derivation at p

$$\partial : \mathcal{O}(X) \to K$$

k-linear

$$\partial(fg) = f(p)\,\partial g + g(p)\,\partial f \qquad (\Rightarrow \partial(c) = 0)$$

$f, g \in m_p \Rightarrow \partial(fg) = 0$ $\quad \partial$ vanishes on m_p^2

also on K

$$\mathcal{O}(X)/m_p^2 \cong K \oplus m_p/m_p^2$$

∂ is essentially a linear functional on m_p/m_p^2 $\leftarrow$ basis $x_1, \dots, x_n$

Get a basis of $T_p X$ (= tangent space) $\partial x_1, \dots, \partial x_n$
dual to $x_1, \dots, x_n$

Also m_p / m_p^2 is $T_p^* X$ (= cotangent space)
(dx_i)

If $\xi : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ is a K-linear derivation
$\xi(fg) = f\,\xi(g) + g\,\xi(f)$

then $f \mapsto (\xi f)(p)$ is a point derivation at p.

So ξ defines a <u>vector field</u> : X smooth $\Rightarrow$ $\{\text{vector fields}\} = \mathrm{Der}_K\, \mathcal{O}(X)$

vector fields = sections of TX (rank $d = \dim X$ vector bundles)

it's a locally free $\mathcal{O}(X)$ module of rank $\dim X$, with basis ∂x_i in reg. coords on a nbhd of p.

T^*X dual of TX, $\quad T_p^*X = m_p/m_p^2$

sections of T^*X are Kahler 1-forms Ω^1 : $\mathcal{O}(X)$-module

with generators df $\quad f \in \mathcal{O}(X)$, relations : d is k-linear

$$d(fg) = f\,dg + g\,df$$

X smooth: $x_1, \ldots, x_n$ reg. local coordinates $\Rightarrow dx_1, \ldots, dx_n$

local basis of Ω^1 $\qquad T^*X \longrightarrow T_p^*X$

$$dx_i \longmapsto \overline{x_i} \in m_p/m_p^2$$

$\mathrm{Der}_k(\mathcal{O}(X))$ is a Lie algebra with $[\,,\,]$ = commutator

$$\begin{aligned}
[\xi, \psi](fg) &= \xi\big(f\psi(g) + g\psi(f)\big) - \psi\big(f\xi(g) + g\xi(f)\big) \\
&= f\,\xi\psi(g) + \xi(f)\psi(g)^{\nearrow 0} + g\,\xi\psi(f) + \xi(g)\psi(f)^{\nearrow 0} \\
&\quad - (\quad \text{same with } \xi \longleftrightarrow \psi \quad) \\
&= f\,[\xi,\psi](g) + g\,[\xi,\psi](f).
\end{aligned}$$

Left invariant vector fields on group G form a Lie subalgebra

$\mathfrak{g} \subseteq \mathrm{Der}_k(\mathcal{O}(X))$ $\qquad$ $\mathfrak{g} \xrightarrow{\sim} T_e X$

G = alg. gp. $/k$ $\qquad$ $A = \mathcal{O}(G)$ $\qquad$ [Hopf alg]

$\xi \in T_e G = \mathfrak{g}$ is in A^*, $\quad A^*$ is an algebra, acts on any right A-comodule V by $V \xrightarrow{\rho} V \otimes A \xrightarrow{id \otimes \xi} V \otimes k = V$

$\xi \in A^*$

$ev(g) \in A^*$ acts as $g \in G$ $\qquad$ $G \curvearrowright V$

Now we get $\mathfrak{g} \curvearrowright V$. In particular, A is an A-comodule via Δ

A^* acts on A by $\xi \in A^*$ $\qquad$ $f \in A$ $\qquad$ $\Delta f = \sum f_{(1)} \otimes f_{(2)}$

$$\xi f = \sum f_{(1)} \, \xi(f_{(2)})$$

left invariant

$$f(g \cdot h) = \sum f_{(1)}(g) \, f_{(2)}(h)$$

$\xi \in \mathfrak{g}$ acts on A as the derivation $\tilde{\xi}$ s.t. $(\tilde{\xi} f)(e) = \xi f$

left invariant der.

(Exercise)

$$(\tilde{\xi} f)(e) = \sum f_{(1)}(e)\, \xi f_{(2)}$$

$$= \xi f .$$

ev_e is the counit ε

$\sum f_{(1)}(e)\, f_{(2)} = f$

$\mathfrak{g} = (m_e / m_e^2)^* \hookrightarrow A^*$ s.t. image closed under commutator

$\rightarrow [\,,\,]$.

$\sum f_{(1)} \otimes f_{(2)}$

$\xi, \psi \in \mathfrak{g}$ $\quad [\xi, \psi] f = \sum \big(\xi(f_{(1)})\, \psi(f_{(2)}) - \psi(f_{(1)})\, \xi(f_{(2)}) \big)$

$(A/m_e^n)^* \hookrightarrow A^*$ gives $\mathcal{U} = \varinjlim_n (A/m_e^n)^* \subset A^*$

$\mathcal{U}$ has a basis $\underline{\partial x}^{(a)}$ dual to $\underline{x}^a$ $\qquad \underline{\partial x}^{(a)} \qquad \frac{\underline{\partial x}^a}{a!}$

$\mathcal{U}$ = left invariant differential ops on A

$\frac{\partial x^m}{m!}(x^n)\big|_0 = \delta_{m,n}$

in char 0, $\mathfrak{g}$ generates $\mathcal{U} = \mathcal{U}(\mathfrak{g})$

in char p: $\partial x_i^{\,p} = 0$. $\mathcal{U}$ is generated by

$\partial x, \ \partial x^{(p)}, \ \partial x^{(p^2)}, \dots$ for $x \in \mathfrak{g}$

$\mathfrak{g}$: $[\,,\,]$ commutator in A^*

$\mathrm{Ad}: G \curvearrowright \mathfrak{g}$ via $G \curvearrowright G$ by conjugation (e is fixed)

Exercise In A^* : $G \subset A^*$ $\quad g \mapsto ev_g$

$\mathfrak{g} = m_e/m_e^2$

$G \curvearrowright \mathfrak{g}$ is conjugation by g "ev_g" in A^*.

Ex. $G = GL_n$ Local coordinates $(X - I)_{ij} \longleftrightarrow \partial X_{ij}$ basis of $\mathfrak{g}$

Compute $[\partial X_{ij}, \partial X_{kl}]$ on f $\quad f(X \cdot X')$

$(\partial X_{ij}\, \partial X'_{kl} - \partial X_{kl}\, \partial X'_{ij})\, f(X \cdot X')$

$f = X_{rs}$ $\quad f(X \cdot X') = \sum_m X_{rm} X'_{ms}$

$\partial X_{ij}\, \partial X'_{kl} - \partial X_{kl}\, \partial X'_{ij} = (1 \text{ if } ij, kl = rm, ms) - (1 \text{ if } ij, kl = ms, rm)$

$(E_{ij} E_{kl})_{rs}$ $\quad (E_{kl} E_{ij})_{rs}$

$$E_{ij} \quad \begin{array}{cc} & j \\ \left(\begin{array}{cc} & 1 \\ 0 & \end{array} \right) & \leftarrow i \end{array}$$

$M_n \ni E_{ij} \longleftrightarrow \partial X_{ij} \in \mathfrak{g}$

$[\ ,\ ] \longleftrightarrow [\ ,\ ]$

$\mathfrak{g} =_{\text{def}} \mathfrak{gl}_n \cong M_n,\ [\ ,\ ]$ $\qquad \partial X_{ij} \longleftrightarrow E_{ij}$

$G \curvearrowright \mathfrak{gl}_n$ $\qquad (\mathrm{Ad}\, g)\ \partial X_{ij}$ $\qquad f \mapsto \partial X_{ij}|_e f(g X g^{-1})$

$\uparrow$

If $f = X_{r,s}$

$\partial X_{ij}|_e \sum_{k,l} g_{rk} X_{kl} (g^{-1})_{ls}$

is $G \curvearrowright M_n$ by conjugation.

$= g_{ri} (g^{-1}_{js})$

$= (g E_{ij} g^{-1})_{r,s}$

(upper uni Δ'ar)

$R_u(B) = U$

$T \hookrightarrow B$

$T \xrightarrow{\sim} B/U$

$B = T \ltimes U$

T is the reductive quotient of B.

Pick $G \supset B \supset T$

$G = GL_n$ $\qquad$ B: Borel = max'l solvable $\qquad$ T: max'l alg torus

$G = GL_n$

$B =$ (upper Δ'ar)

$T =$ (diagonal matrix)

$\mathrm{Ad} : T \curvearrowright \mathfrak{g}$